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A REAL GENERALIZATION OF THE DASS-GUPTA FIXED POINT THEOREM

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Abstract. In this paper, we investigate the existence of a unique fixed point of self-mappings satisfying the dualistic Dass-Gupta contractions defined on a complete dualistic partial metric space under the influence of convergence comparison property. As an application, we demonstrate the existence and uniqueness of the solution of functional equations in dynamic programming.

Keywords: fixed point, dualistic partial metric spaces, functional equations.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

Matthews [12] introduced the partial metric space and generalized the Banach fixed point theorem. The notions such as convergence, completeness, Cauchy sequence in the setting of partial metric spaces, can be found in [1-6, 9-11] and references there in. On the other hand, Neill [16] did one significant change to the definition of the partial metric by extending its range from $[0, \infty)$ to $(-\infty, \infty) = \mathbb{R}$. The partial metric with this extended range is known as a dualistic partial metric. Oltra *et al.* [15] established the criteria for convergence of sequences and completeness in the dualistic partial metric spaces and generalized the fixed point result presented by Matthews.

In this paper, we shall consider the fixed point theorem of the Dass-Gupta type (Theorem 1.1 below) and investigate its validity in dualistic partial metric spaces (Theorem 3.1 given below). We claim that our work is a real generalization of Theorem 1.1. Recently in [14, 17] established some new fixed point theorems on dualistic rational contraction.

Dass-Gupta [8] presented the following fixed point theorem.

Theorem 1.1. *Let $T : X \to X$ be a self-mapping defined on a complete metric space (X, d). Suppose there exista $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that*

$$d(T(x), T(y)) \leq \frac{\alpha d(y, T(y))(1 + d(x, T(x)))}{1 + d(x, y)} + \beta d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

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2. Preliminaries

Matthews [12] generalized the notion of a metric as follows:

Definition 2.1. [12], *Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ satisfying the following axioms:*

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \leq p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for all $x, y, z \in X$,

is called a partial metric on X. The pair (X, p) is called a partial metric space.

Neill [16] changed the definition of the partial metric p by considering its range as $(-\infty, \infty) = \mathbb{R}$. The modified partial metric p with extended range is known as a dualistic partial metric. We denote it as p^*.

Definition 2.2. [16], *Let X be a nonempty set. The mapping $p^* : X \times X \to \mathbb{R}$ satisfying, for all $x, y, z \in X$, the following axioms:*

(p_1^*) $x = y \Leftrightarrow p^*(x, x) = p^*(y, y) = p^*(x, y)$;
(p_2^*) $p^*(x, x) \leq p^*(x, y)$;
(p_3^*) $p^*(x, y) = p^*(y, x)$;
(p_4^*) $p^*(x, z) + p^*(y, y) \leq p^*(x, y) + p^*(y, z)$,

is called a dualistic partial metric and the pair (X, p^*) is known as a dualistic partial metric space.

If (X, p^*) is a dualistic partial metric space, then mapping $d_{p^*} : X \times X \to \mathbb{R}^+$ defined by

$$d_{p^*}(x, y) = p^*(x, y) - p^*(x, x), \text{ for all } x, y \in X. \tag{1}$$

is called an induced quasi-metric on X such that $\tau(p^*) = \tau(d_{p^*})$. Moreover, $d^s_{p^*}(x, y) = \max\{d_{p^*}(x, y), d_{p^*}(y, x)\}$ defines a metric on X (the induced metric).

Remark 2.1. *Unlike other metrics, in dualistic partial metric $p^*(x, y) = 0$ does not imply $x = y$. Indeed, $p^*(-1, 0) = 0$ and $0 \neq -1$. This situation creates a problem in obtaining a fixed point of a self-mapping in dualistic partial metric space. For the solution of this problem we introduce convergence comparison property (defined below) and use it along with axioms (p_2^*) and (p_1^*) to get a fixed point.*

Example 2.1. *Define the mapping $p^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$* by $p^*(x_1, x_2) = \max\{x_1, x_2\}$. *It is easy to check that p^* satisfies $(p_1^*) - (p_4^*)$ and hence p^* is a dualistic partial metric on $\mathbb{R}$. Note that p^* does not define a partial metric on $\mathbb{R}$ due to the fact that $p^*(-m, -m) = -m$ for $m > 0$.*

Example 2.2. *Let (X, p) be a partial metric space. The mapping $p^* : X \times X \to \mathbb{R}$ defined by*

$$p^*(x, y) = p(x, y) - p(x, x) - p(y, y) \text{ for all } x, y \in X,$$

satisfies the conditions $(p_1^*) - (p_4^*)$ and hence defines a dualistic partial metric on X. We note that $p^*(x, y)$ may have negative values.

Example 2.3. *Let $X = \mathbb{R}$. Define the mapping $p^* : X \times X \to \mathbb{R}$ by*

$$p^*(x, y) = \begin{cases} |x - y|, & \text{if } x \neq y, \\ -b, & \text{if } x = y; b > 0. \end{cases}$$

The axioms (p_1^*), (p_2^*) and (p_3^*) can be proved immediately. We prove axiom (p_4^*) in details. *If $x \neq y = z$, then*
$p^*(x, z) \leq p^*(x, y) + p^*(y, z) - p^*(y, y)$ *implies* $|x - z| = |x - y|$.

If $x = y \neq z$, then
$p^*(x,z) \leq p^*(x,y) + p^*(y,z) - p^*(y,y)$ *implies* $|x-z| = |y-z|$.
If $x = y = z$, then
$p^*(x,z) \leq p^*(x,y) + p^*(y,z) - p^*(y,y)$ *implies* $-b = -b$.
If $x \neq y \neq z$, then
$p^*(x,z) \leq p^*(x,y) + p^*(y,z) - p^*(y,y)$ *implies* $|x-z| \leq |x-y| + |y-z| + b$.

Thus, the axiom (p_4^*) holds in all cases. Hence (X, p^*) is a dualistic partial metric space. Neill [16] established that each dualistic partial metric p^* on X generates a T_0 topology $\tau[p^*]$ on X having bases the family of p^*-balls $\{B_{p^*}(x,\epsilon) \,:\, x \in X, \epsilon > 0\}$ where

$$B_{p^*}(x,\epsilon) = \{y \in X \,:\, p^*(x,y) < \epsilon + p^*(x,x)\}.$$

The following and Lemma describe the convergence criteria established by Oltra *et al.* [15].

Definition 2.2.[15] *Let (X, p^*) be a dualistic partial metric space.*

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p^*) is called a Cauchy sequence if

$$\lim_{n,m\to\infty} p^*(x_n, x_m) \text{ exists and is finite.}$$

(2) A dualistic partial metric space (X, p^*) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p^*]$, to a point $\upsilon \in X$ such that

$$p^*(x,x) = \lim_{n,m\to\infty} p^*(x_n, x_m).$$

Lemma 2.1.[15] *Let (X, p^*) be a dualistic partial metric space.*

(1) Every Cauchy sequence in $(X, d^s_{p^*})$ is also a Cauchy sequence in (X, p^*).
(2) A dualistic partial metric (X, p^*) is complete if and only if the induced metric space $(X, d^s_{p^*})$ is complete.
(3) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d^s_{p^*})]$ if and only if

$$\lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon,\upsilon) = \lim_{n\to\infty} p^*(x_n, x_m).$$

3. Main results

Dass-Gupta[8] have employed a rational type contractive condition on T to find a unique fixed point of T in the context of a metric space. We introduce the convergence comparison property [in short: CCP] and impose it to find the fixed point of a self-mapping T satisfying the dualistic Dass-Gupta contractive condition (defined below) in a complete dualistic partial metric space.

Definition 3.1. *Let (X, p^*) be a dualistic partial metric space and $T : X \to X$ be a mapping. We say that T has a convergence comparison property (CCP) if for every sequence $\{x_n\}$ in X such that $x_n \to x$, T satisfies*

$$p^*(x,x) \leq p^*(T(x), T(x)).$$

Example 3.1.*Let $X = \mathbb{R}$. Define $p^*_\vee : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by*

$$p^*_\vee(x,y) = \max\{x,y\} \text{ for all } x, y \in \mathbb{R}.$$

It is easy to check that $(X, p^*_\vee)$ is a dualistic partial metric space. Consider any sequence $\{x_n\}$ converging to x in $(X, p^*_\vee)$. Define $T : X \to X$ by $T(x) = e^x$. We have $x \leq e^x$ for any $x \in X$, that is, $p^*_\vee(x,x) \leq p^*_\vee(T(x), T(x))$, i.e., T satisfies (CCP).

Definition 3.2. *Let $T : X \to X$ be a self-mapping define on a dualistic partial metric space $(X \neq \phi, p^*)$. We say the mapping T is a dualistic Dass-Gupta contraction, if there exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that*

$$|p^*(T(x), T(y))| \leq \left|\frac{\alpha p^*(y, T(y))(1 + p^*(x, T(x)))}{1 + p^*(x, y)}\right| + \beta|p^*(x, y)|, \tag{2}$$

for all $x, y \in X$.

Theorem 3.1. *Let T be a dualistic Dass-Gupta contraction defined on a complete dualistic partial metric space (X, p^*). If T is continuous and satisfies (CCP), then T has a unique fixed point in X and the Picard iterative sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ with initial point x_0, converges to the fixed point.*

Proof. Let x_0 be an initial point of X and define Picard iterative sequence $\{x_n\}$ by

$$x_n = T(x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer i such that $x_i = x_{i+1}$, then $x_i = x_{i+1} = T(x_i)$, so x_i is a fixed point of T. In this case, proof is finished. Now, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then by (2), we have

$$\begin{aligned}
|p^*(T(x_{n-1}), T(x_n))| &\leq \left|\frac{\alpha p^*(x_n, T(x_n))(1 + p^*(x_{n-1}, T(x_{n-1})))}{1 + p^*(x_{n-1}, x_n)}\right| + \beta|p^*(x_{n-1}, x_n)| \\
&\leq \left|\frac{\alpha p^*(x_n, x_{n+1})(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_n)}\right| + \beta|p^*(x_{n-1}, x_n)| \\
&\leq |\alpha p^*(x_n, x_{n+1})| + \beta|p^*(x_{n-1}, x_n)| \\
&\leq \frac{\beta}{1-\alpha}|p^*(x_{n-1}, x_n)|.
\end{aligned}$$

If we set $\lambda = \dfrac{\beta}{1-\alpha}$, then $0 < \lambda < 1$, and so

$$\begin{aligned}
|p^*(x_n, x_{n+1})| &\leq \lambda|p^*(x_{n-1}, x_n)| \\
&\leq \lambda(\lambda|p^*(x_{n-2}, x_{n-1})|) \\
&\ \ \vdots \\
&\leq \lambda^n|p^*(x_0, x_1)|.
\end{aligned}$$

This implies that

$$|p^*(x_{n+k-1}, x_{n+k})| \leq \lambda^{n+k-1}|p^*(x_0, x_1)|, \quad \text{for all } n, k \in \mathbb{N}. \tag{3}$$

Again by (2), we have

$$\begin{aligned}
|p^*(T(x_0), T(x_0))| &\leq \left|\frac{\alpha p^*(x_0, T(x_0))(1 + p^*(x_0, T(x_0)))}{1 + p^*(x_0, x_0)}\right| + \beta|p^*(x_0, x_0)| \\
&\leq \left|\frac{\alpha p^*(x_0, x_1)(1 + p^*(x_0, x_1))}{1 + p^*(x_0, x_0)}\right| + \beta|p^*(x_0, x_0)| \\
&\leq |\alpha p^*(x_0, x_1)(1 + p^*(x_0, x_1))| + \beta|p^*(x_0, x_0)|,
\end{aligned}$$

that is,

$$|p^*(x_1, x_1)| \leq \alpha h(1 + h) + \beta\kappa.$$

where we have set $|p^*(x_0,x_1)| = h$ and $|p^*(x_0,x_0)| = \kappa$. Similarly,

$$\begin{aligned}
|p^*(x_2,x_2)| &\leq \alpha\lambda h(1+\lambda h) + \alpha\beta h(1+h) + \beta^2\kappa,\\
|p^*(x_3,x_3)| &\leq \alpha\lambda^2 h(1+\lambda^2 h) + \alpha\beta\lambda h(1+\lambda h) + \alpha\beta^2 h(1+h) + \beta^3\kappa,\\
&\vdots\\
|p^*(x_n,x_n)| &\leq \alpha\lambda^{n-1}h(1+\lambda^{n-1}h) + \alpha\beta\lambda^{n-2}h(1+\lambda^{n-2}h)\\
&+ \alpha\beta^2\lambda^{n-3}h(1+\lambda^{n-3}h) + \cdots + \alpha\beta^{n-1}h(1+h) + \beta^n\kappa.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} p^*(x_n,x_n) = 0. \tag{4}$$

By definition of d_{p^*} and (3), we have

$$d_{p^*}(x_n,x_{n+1}) \leq |p^*(x_n,x_{n+1})| + |p^*(x_n,x_n)|,$$

that is,

$$\begin{aligned}
d_{p^*}(x_n,x_{n+1})) &\leq \lambda^n|p^*(x_0,x_1)| + |p^*(x_n,x_n)|\\
&\leq \lambda^n h + \alpha\lambda^{n-1}h(1+\lambda^{n-1}h) + \alpha\beta\lambda^{n-2}h(1+\lambda^{n-2}h)\\
&+ \alpha\beta^2\lambda^{n-3}h(1+\lambda^{n-3}h) + \cdots + \alpha\beta^{n-1}h(1+h) + \beta^n\kappa\\
&\leq \lambda^n h + \mu^n,
\end{aligned}$$

where $\mu^n = \alpha\lambda^{n-1}h(1+\lambda^{n-1}h) + \alpha\beta\lambda^{n-2}h(1+\lambda^{n-2}h) + \alpha\beta^2\lambda^{n-3}h(1+\lambda^{n-3}h)$
$+\cdots+\alpha\beta^{n-1}h(1+h)+\beta^n\kappa$. Moreover, we have

$$d_{p^*}(x_{n+k-1},x_{n+k}) \leq \lambda^{n+k-1}h + \mu^{n+k-1}, \text{ for all } n,k\in\mathbb{N}. \tag{5}$$

We show that $\{x_n\}$ is Cauchy sequence in $(X,d^s_{p^*})$. By triangle property and (5), we have

$$\begin{aligned}
d_{p^*}(x_n,x_{n+k}) &\leq d_{p^*}(x_n,x_{n+1}) + d_{p^*}(x_{n+1},x_{n+2}) + \cdots + d_{p^*}(x_{n+k-1},x_{n+k})\\
&\leq \lambda^n h + \mu^n + \lambda^{n+1}h + \mu^{n+1} + \cdots + \lambda^{n+k-1}h + \mu^{n+k-1}\\
&\leq \frac{\lambda^n}{1-\lambda}h + \frac{\mu^n}{1-\mu}.
\end{aligned}$$

Since $\lim_{n\to\infty}\lambda^n = 0$ and $\lim_{n\to\infty}\mu^n = 0$, we get

$$\lim_{n\to\infty} d_{p^*}(x_n,x_{n+k}) = 0. \tag{6}$$

Similarly,

$$\begin{aligned}
d_{p^*}(x_{n+1},x_n)) &\leq \lambda^n|p^*(x_0,x_1)| + |p^*(x_{n+1},x_{n+1})|\\
&\leq \lambda^n h + \mu^{n+1}.
\end{aligned}$$

Thus,

$$d_{p^*}(x_{n+k},x_{n+k-1}) \leq \lambda^{n+k-1}h + \mu^{n+k} \text{ for all } n,k\in\mathbb{N}. \tag{7}$$

By triangle property and (7), we have

$$\begin{aligned}
d_{p^*}(x_{n+k},x_n) &\leq d_{p^*}(x_{n+k},x_{n+k-1}) + d_{p^*}(x_{n+k-1},x_{n+k-2}) + \cdots + d_{p^*}(x_{n+1},x_n)\\
&\leq \lambda^{n+k-1}h + \mu^{n+k} + \lambda^{n+k-2}h + \mu^{n+k-1} + \cdots + \lambda^n h + \mu^{n+1}\\
&\leq \frac{\lambda^n}{1-\lambda}h + \frac{\mu^{n+1}}{1-\mu}.
\end{aligned}$$

Since $\lim_{n\to\infty}\lambda^n = 0$ and $\lim_{n\to\infty}\mu^n = 0$, we have $\lim_{n\to\infty} d_{p^*}(x_{n+k},x_n) = 0$.
Hence, $\lim_{n\to\infty} d^s_{p^*}(x_n,x_{n+k}) = 0$. This shows that $\{x_n\}$ is a Cauchy sequence in $(X,d^s_{p^*})$. Since

$(X, d^s_{p^*})$ is a complete metric space, therefore $\{x_n\}$ converges to a point in X, say υ, that is $\lim_{n\to\infty} d^s_{p^*}(x_n, \upsilon) = 0$. By Lemma 2.1, one writes

$$\lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon, \upsilon) = \lim_{n,m\to\infty} p^*(x_n, x_m). \tag{8}$$

By (4) and (6), we have

$$\lim_{n,m\to\infty} p^*(x_n, x_m) = \lim_{n\to\infty} p^*(x_n, x_n) = 0;\ m = n + k.$$

Consequently, $\{x_n\}$ is Cauchy sequence in (X, p^*). By (8), we have

$$\lim_{n\to\infty} p^*\left(\upsilon, x_n\right) = p^*(\upsilon, \upsilon) = 0. \tag{9}$$

This shows that $\{x_n\}$ converges to υ in (X, p^*). By (2), we get

$$\begin{aligned} |p^*(x_{n+1}, T(\upsilon))| &\leq \left|\frac{\alpha p^*(\upsilon, T(\upsilon))(1 + p^*(x_n, T(x_n)))}{1 + p^*(x_n, \upsilon)}\right| + \beta|p^*(x_n, \upsilon)| \\ &\leq \left|\frac{\alpha p^*(\upsilon, T(\upsilon))(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_n, \upsilon)}\right| + \beta|p^*(x_n, \upsilon)|. \end{aligned}$$

As $n \to \infty$, we get $p^*(\upsilon, T(\upsilon)) = 0$. Since T has (CCP),

$$0 = p^*(\upsilon, \upsilon) \leq p^*(T(\upsilon), T(\upsilon)).$$

By (p^*_2), we deduce that

$$p^*(T(\upsilon), T(\upsilon)) \leq p^*(\upsilon, T(\upsilon)) = 0.$$

This shows that $p^*(T(\upsilon), T(\upsilon)) = 0$. Thus,

$$p^*(T(\upsilon), T(\upsilon)) = p^*(\upsilon, \upsilon) = p^*(\upsilon, T(\upsilon)),$$

and axiom (p^*_1) leads us to conclude that $\upsilon = T(\upsilon)$ and hence υ is a fixed point of T.
To prove the uniqueness, suppose that ω is another fixed point of T, then $T(\omega) = \omega$ and $D(\omega, \omega) = 0$. Using (2), we obtain

$$\begin{aligned} |p^*(\upsilon, \omega)| = |p^*(T(\upsilon), T(\omega))| &\leq \left|\frac{\alpha p^*(\omega, T(\omega))(1 + p^*(\upsilon, T(\upsilon)))}{1 + p^*(\upsilon, \omega)}\right| + \beta|p^*(\upsilon, \omega)|. \\ &\leq \left|\frac{\alpha p^*(\omega, T(\omega))(1 + p^*(\upsilon, T(\upsilon)))}{1 + p^*(\upsilon, \omega)}\right| + \beta|p^*(\upsilon, \omega)| \\ &\leq \beta|p^*(\upsilon, \omega)|, \end{aligned}$$

This implies that $\upsilon = \omega$, which proves the uniqueness of υ. □

Corollary 3.1.[15, Theorem 2.3]; *Let (X, p^*) be a complete dual partial metric space and $T : X \to X$ be a continuous mapping. Suppose there exists $\beta \in [0, 1)$ such that*

$$|p^*(T(x), T(y))| \leq \beta|p^*(x, y)|,$$

for all $x, y \in X$. Then T has a unique fixed point υ. Moreover, $p^*(\upsilon, \upsilon) = 0$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to υ with respect to $\tau(d^s_{p^*})$ for every $x \in X$.

Proof. Set $\alpha = 0$ in Theorem 3.1. □

Corollary 3.2.[13] *Let $T : X \to X$ be a self-mapping on a complete partial metric space (X, p). Assume there exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that*

$$p(T(x), T(y)) \leq \frac{\alpha p(y, T(y))(1 + p(x, T(x)))}{1 + p(x, y)} + \beta p(x, y),$$

for all $x, y \in X$. Then T has a fixed point in X.

Proof. Since the restriction of a dualistic partial metric p^* to $\mathbb{R}_0^+$, which is $p^*|_{\mathbb{R}_0^+} = p$, is a partial metric, arguments follow the same lines as in the proof of Theorem 3.1. $\square$

The following example elucidates our results.

Example 3.1. *Let $X = \mathbb{R}$. Define $p_\vee^* : X \times X \to \mathbb{R}$ by $p_\vee^*(x,y) = x \vee y$, for all $x, y \in X$. Note that $(X, p_\vee^*)$ is a complete dualistic partial metric space. Let $T : X \to X$ be given by*

$$T(x) = \frac{x}{2} \text{ for all } x \in X.$$

Clearly, T is a continuous mapping. We shall show that for all $x, y \in X$, (2) is satisfied. Without loss of generality, we can assume that $x \geq y$. Note that

$$|p_\vee^*(T(x), T(y))| \leq \left| \frac{\alpha p_\vee^*(y, T(y))(1 + p_\vee^*(x, T(x)))}{1 + p_\vee^*(x,y)} \right| + \beta |p_\vee^*(x,y)|,$$

is equivalent to

$$\left| p_\vee^*(\frac{x}{2}, \frac{y}{2}) \right| \leq \left| \frac{\alpha p_\vee^*(y, \frac{y}{2})(1 + p_\vee^*(x, \frac{x}{2}))}{1 + p_\vee^*(x,y)} \right| + \beta |p_\vee^*(x,y)|. \tag{10}$$

Take $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$. We have the following cases:
If $x, y \geq 0$, then (10) is equivalent to $|\frac{x}{2}| \leq \alpha |y| + \beta |x|$, which holds.
If $x, y < 0$, then (10) is equivalent to $|\frac{x}{2}| \leq \alpha |\frac{y}{2}| + \beta |x|$, which is satisfied.
If $x > 0$ and $y < 0$, then (10) is equivalent to $|\frac{x}{2}| \leq \alpha |\frac{y}{2}| + \beta |x|$, which is verified.
Thus, all the conditions of Theorem 3.1 are satisfied. Moreover, $x = 0$ is the unique fixed point of T.
The following example emphasizes the use of absolute value functions in the contractive condition (2).

Example 3.2. *Let $X = (-1, 0]$* and *$p_\vee^* : X \times X \to \mathbb{R}$ be defined as $p_\vee^*(x,y) = x \vee y$,* for all *$x, y \in X$. Note that $(X, p_\vee^*)$ is a complete dual partial metric space. Let $T : X \to X$ be given by*

$$T(x) = \frac{x}{2} \text{ for all } x \in X.$$

The inequality

$$\left| \frac{x}{2} \right| \leq \alpha \left| \frac{y(x+2)}{4(x+1)} \right| + \beta |x|,$$

holds for all $x, y \in X$ and for some choice of α, β such that *$\alpha + \beta < 1$. Thus, the contractive condition*

$$|p_\vee^*(T(x), T(y))| \leq \left| \frac{\alpha p_\vee^*(y, T(y))(1 + p_\vee^*(x, T(x)))}{1 + p_\vee^*(x,y)} \right| + \beta |p_\vee^*(x,y)|,$$

is satisfied and $x = 0$ is the unique fixed point of T. However, note that for $x = -\frac{3}{4}$, $y = -\frac{1}{2}$, $\alpha = \frac{2}{5}$ and $\beta = \frac{1}{3}$, the contractive condition

$$p_\vee^*(T(x), T(y)) \leq \frac{\alpha p_\vee^*(y, T(y))(1 + p_\vee^*(x, T(x)))}{1 + p_\vee^*(x,y)} + \beta p_\vee^*(x,y),$$

does not hold. Hence Corollary 3.2 is not applicable.

4. Application

As an application of Theorem 3.1, we show the existence and uniqueness of the solution of a functional equation. First, we introduce some notations for the sake of convenience.

$$S = \text{State space}, \qquad W = \text{Decision space},$$

$$B(S) = \text{Space of bounded functions}, g : S \times W \to \mathbb{R},$$

$$F : S \times W \times \mathbb{R} \to \mathbb{R}, \qquad \phi : S \times W \to S.$$

We shall prove the existence and uniqueness of the solution of a functional equation appearing in dynamic programming (for example, see [7]):

$$u(x) = \sup_{y\in W}\{g(x,y) + F(x,y,u(\phi(x,y)))\}, \text{ for all } x \in S. \tag{11}$$

We observe that the space $(B(S), \|.\|_\infty)$ endowed with the norm defined by

$$d(u,v) = \|u - v\|_\infty = \sup_{x\in S}|u(x) - v(x)| \text{ for all } u,v \in B(S),$$

is a Banach space. Given the dualistic metric as

$$p^*(u,v) = d(u,v) - c, \text{ for all } u,v \in B(S),$$

where $c > 0$.

Lemma 4.1.[7] *Let $G, H : S \to \mathbb{R}$ be two bounded functions. Then*

$$|\sup_{x\in S} G(x) - \sup_{x\in S} H(x)| \le \sup_{x\in S}|G(x) - H(x)|.$$

Lemma 4.2. [7] *Assume that*

(1) *g and F are bounded functions;*

(2) *there exists $k > 0$ such that for all $t, r \in \mathbb{R}$, $x \in S$ and $y \in W$,*

$$|F(x,y,t) - F(x,y,r)| \le k|t - r|.$$

Then the operator $R : B(S) \to B(S)$ defined by

$$(Ru)(x) = \sup_{y\in W}\{g(x,y) + F(x,y,u(\phi(x,y)))\},$$

is well defined.

Now, we present the main result.

Theorem 4.1. Assume that (1) and (2) in Lemma 4.2 are satisfied. Suppose that

$$\sup_{y\in W}|F(x,y,u) - F(x,y,v)| + c \le \left|\frac{\alpha p^*(u,Ru)(1 + p^*(v,Rv))}{1 + p^*(u,v)}\right| + \beta|p^*(u,v)|. \tag{12}$$

Then the functional equation (11) has a unique solution.

Proof. Let $R : B(S) \to B(S)$ be an operator defined as in Lemma 4.2. We show that R satisfies the contractive condition (2). By Lemma 4.1, for $u, v \in B(S)$.

$$\begin{aligned}
& |(Ru)(x) - (Rv)(x)| \\
= \; & |\sup_{y\in W}\{g(x,y) + F(x,y,u(\phi(x,y)))\} - \sup_{y\in W}\{g(x,y) + F(x,y,v(\phi(x,y)))\}| \\
\le \; & \sup_{y\in W}|g(x,y) + F(x,y,u(\phi(x,y))) - g(x,y) - F(x,y,v(\phi(x,y)))| \\
\le \; & \sup_{y\in W}|F(x,y,u(\phi(x,y))) - F(x,y,v(\phi(x,y)))|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|p^*(Ru, Rv)| &= |\sup_{x\in S} |(Ru)(x) - (Rv)(x)| - c| \\
&\le \sup_{x\in S} |(Ru)(x) - (Rv)(x)| + c \\
&\le \sup_{y\in W} |F(x, y, u(\phi(x, y))) - F(x, y, v(\phi(x, y)))| + c.
\end{aligned}$$

We get using (12),

$$|p^*(Ru, Rv)| \le \left| \frac{\alpha p^*(u, Ru)(1 + p^*(v, Rv))}{1 + p^*(u, v)} \right| + \beta |p^*(u, v)|.$$

Hence, R satisfies all conditions of Theorem 3.1, so there exists a unique solution of (11), say $u_0 \in B(S)$ such that $Ru_0 = u_0$. □

5. Conclusions

The unique fixed point of self-mapping satisfying the Dass-Gupta type contraction can be obtained under the effect of the dualistic partial metric, if we assume Convergence Comparison Property in Theorem 3.1.

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